Pergamon

Int. J. Solids Structures Vol. 31, No. 5, pp. 683–704, 1994
Elsevier Science Ltd
Printed in Great Britain
0020–7683/94 $6.00 + .00

VARIATIONAL MICRO–MACRO TRANSITION, WITH APPLICATION TO REINFORCED MORTARS

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(*Received* 13 *October* 1992; *in revised form* 26 *March* 1993)

Abstract—A variational approach is developed for the micro–macro transition in non-linear and randomly inhomogeneous materials, assuming a convex potential and a no-correlation condition. To establish the latter in the context of operational micro–macro models, an asymptotic definition of statistically homogeneous (S.H.) materials and S.H. micro-fields was given; also, a variational model was proposed for S.H. materials with convex local potential, taking into account the volume fractions of the "states" and the average inhomogeneity r of the local stimulus. This statistical theory and this variational model are summarized here. A principle of minimal inhomogeneity is found to underly the success of the model. The "state" contains the information considered relevant on the local behavior and micro-geometry. The approach is illustrated by its application to the failure criterion of a fibre-reinforced mortar. Two successive definitions of the state lead to (i) a volume-fraction model of the composite and (ii) a model accounting for the interaction between neighboring constituents. Model (ii) makes use of the homogenization theory for periodic media and restrains strongly the distance between the upper and lower bounds. For the studied composite, model (i) is yet found to give as good agreement as model (ii), due to the oversimplified micro-structural information entered in model (ii).

1. INTRODUCTION

The search for relationships between local and global properties of materials has long been an important task in academic science, since it often allows a deeper understanding of physical processes. More recently, it has also become important for practical purposes, namely the design of new materials as well as the optimization of their industrial processing. This is due to, and still motivates, the emergence of new procedures for relating local and global behaviors with efficiency and precision. There are two situations in which satisfactory procedures already exist: (i) the material can be described as spatially periodic and (ii) the local behavior may be considered as linear and time-independent.

The first case is handled by the homogenization theory for periodic media (HTPM) proposed by Sanchez-Palencia (1974) and developed, among others, by him (Sanchez-Palencia, 1980), by Bensoussan *et al.* (1978) and (especially for the non-linear behavior which is relevant in the mechanics of materials) by Suquet (1982, 1987). In the second situation, the global linear behavior may be approached from the knowledge of certain statistics of the distribution of the local tensor which characterizes the local linear behavior, by using the theory of Kröner (1972, 1986): the more complete a statistical knowledge one gets, the more exact approximation is obtained. Yet the extension to non-linear behavior is problematic, even though attempts have been made to use Kröner's approach so as to obtain refined self-consistent models accounting for spatial correlations (Zaoui, 1987). The original self-consistent models assume a state of "perfect disorder" [see Zaoui (1987) for an analysis of these models, including many references]; this is the idealized case where no spatial correlation exists, in the sense that the knowledge of the volume fractions (the probability density of order 1 of the distribution of the local tensor) also determines the probability densities of the higher order (Kröner, 1986). Rather successful extensions of the self-consistent approach have been proposed by Molinari *et al.* (1987), and further developed by Adams and Field (1991) whose work is consistently based on statistical continuum mechanics. They incorporate some elements of the spatial distribution of a local

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tensor (which characterizes the assumed quasi-linear, viscoplastic behavior) as secondary contributions in an integral equation—the latter resulting from a formulation of the micro–macro mechanical problem with the Green function method. Despite their success in producing less sharp predictions, these approaches are very complex and involve mathematical and numerical difficulties as well as various closure assumptions. Moreover they do not seem to be easy to adapt to different homogenization problems in materials with a random structure.

This paper has two main objectives: first, to present and to develop a statistical theory and a variational model proposed by Arminjon (1991a), in a way that should make their essential content and their practical operation easier to see. In the original work, the emphasis was more on the generality as well as on the mathematical rigour. The statistical theory gives a new general framework for discussing micro–macro models for randomly inhomogeneous media, which is simpler than the classical approach based on ergodic theory. The variational model is the extension of a previous model to inhomogeneous media of any kind, provided that they fulfil the general requirements of the statistical theory and that their local behavior derives from a convex potential; the earlier version of this model was proposed for plastically deformed polycrystals and successfully applied to them, especially in what regards the question of the deformation textures in steels (Arminjon, 1987; Arminjon and Donadille, 1990). Two important new developments are presented; the first, theoretical one, is the recognition of a principle of minimal inhomogeneity, underlying the proposed variational model. The second one is an effective procedure for taking into account the micro-geometry of the material, based on the homogenization theory for periodic media (HTPM). This procedure offers an alternative to the consideration of correlation functions of increasing order.

The other purpose of this paper is to illustrate the application of the method in the case of mortars reinforced with steel hooks. This material is important in civil engineering. It has the didactic advantage of a relatively simple spatial structure with isotropic constituents, but it is not an easy material for a micro–macro model, because it is highly non-linear and the behavior is very different from one phase (mortar) to another (steel). This, as will be seen, leads to a very poor prediction of the overall behavior if the elementary volume-fraction models are used. It is thus interesting to check the volume-fraction approximation in the proposed model, and to compare this with the results of a more sophisticated application of the model, involving a double-scale micro–macro transition allowing to take micro-geometrical information into account. The latter application is obtained with the important help of the periodic approach (HTPM), as it has been developed for plasticity by Le Nizhery (1976) and Suquet (1982, 1987) and more particularly for the analysis of limit loads by Turgeman and Pastor (1987). It is also emphasized that the rigid–plastic approximation has been used to model the behavior of the material, and this might be considered an important simplification of the observed behavior. However, the safety analysis of civil engineering structures is still based on limit analysis, i.e. on the rigid–plastic idealization. The proposed approach already proves to be predictive, and the calculations would be much more difficult if a more realistic schematization, involving damage and partial localization, was adopted.

2. MICRO–MACRO TRANSITION IN STATISTICALLY HOMOGENEOUS MATERIALS

2.1. *Motivation: some general results*

The main task in effecting a micro–macro transition is deduction of a relationship between macroscopic tensors $\mathbf{S}$ and $\mathbf{R}$ of "stimulus" and "response" from the microscopic constitutive equation connecting the corresponding microscopic tensors $\mathbf{s}$ and $\mathbf{r}$. {This notation seems both general and easy, but in order to fix the ideas one may already substitute the strain-rate $\mathbf{d}$ and the Cauchy stress $\mathbf{t}$ for $\mathbf{s}$ and $\mathbf{r}$, respectively, since the application presented here deals with rigid–perfectly plastic behavior [thus the stress at yielding is $\mathbf{t} = \mathbf{f}(\mathbf{d})$ with $\mathbf{f}(\lambda\mathbf{d}) = \mathbf{f}(\mathbf{d})$ if $\lambda > 0$, see Sawczuk and Stutz (1968)]; the same pair would be used for viscous or viscoplastic behavior. For elastic behavior, one would substitute a strain

and a stress tensor for **s** and **r**, or the reverse; for elastoplastic behavior, one would have the strain-rate (or the velocity gradient) and a stress-rate; etc.} This *homogenization process* often demands that one is able to determine the micro-fields **s** and **r**, depending on the micro-position, **x**, from the mere knowledge of **S** (*localization process*)—a problem which can clearly have a unique solution only if **s** and **r** are subjected to some restrictive, physically motivated constraints. Moreover the localization must be done if the local properties of the material evolve with time as functions of the micro stimulus **s**. As has been observed by Bishop and Hill (1951) and Hill (1952) for the important case of a local stress–strain behavior deriving from a potential u, the key role in the homogenization problem is played by the condition that "admissible" strain and stress micro-fields $\mathbf{s}'$ and **r** should have no macroscopic correlation:

$$\overline{\mathbf{s}' : \mathbf{r}} = \bar{\mathbf{s}}' : \bar{\mathbf{r}} = \mathbf{S}' : \mathbf{R}, \tag{1}$$

where the superior bar denotes volume average, double point means scalar product of tensors (when regarded as vectors in a Euclidean space) and the prime indicates that $\mathbf{s}'$ and **r** are not necessarily "associated" fields, in the sense that they do not need to correspond to one and the same macro-stimulus **S**. Equation (1) enabled Bishop and Hill (1951) to extend Hill's maximum work principle from the constituent crystals to the polycrystal. Later on, Hill (1967, 1984) formulated a sufficient "macro-homogeneity condition" for elastoplastic materials, ensuring that eqn (1) is fulfilled by any relevant pair of micro-fields; his analysis is reviewed and extended by Arminjon (1991b) who proposes a general construction of macro-homogeneous strain fields. Hill (1967, 1984) also deduced from eqn (1) (in the particular case of elastoplastic behavior) that the average potential is a potential for the macroscopic constitutive equation:

$$\mathbf{R} = \frac{\partial U}{\partial \mathbf{S}}, \quad U(\mathbf{S}) = \overline{u(\mathbf{s})} = \frac{1}{V(\Omega)} \int_{\Omega} u(\mathbf{s}(\mathbf{x}), \mathbf{x}) \, \mathrm{d}V(\mathbf{x}). \tag{2}$$

Here **s** is an "effective" micro-field such that $\bar{\mathbf{s}} = \mathbf{S}$, and Ω is a "representative volume element (RVE)" (see Section 2.2). Mandel (1972) obtained similar results for viscoplastic behavior. The stimulus field **s** is effective in the sense that: (i) it is "admissible"; e.g. if **s** is a strain-rate field **d**, it derives from a regular velocity field **v** satisfying the boundary conditions, and moreover **s** is associated by the local constitutive equation with a response field **r**:

$$\mathbf{r}(\mathbf{x}) = \frac{\partial u}{\partial \mathbf{s}}(\mathbf{s}(\mathbf{x}), \mathbf{x}), \tag{3}$$

and (ii) the associated response field **r** is *also* an "admissible" one (if **r** is a stress field **t**, it obeys the equilibrium equations and the boundary conditions and in the case of rigid–plastic behavior, **t** is not outside the yield criterion).

It seems interesting to emphasize the simple and general way in which Hill's essential results, as well as a possible *procedure for solving the homogenization–localization problem*, depend only on the no-correlation condition (1) and on the existence of a potential, assumed *convex*, for the micro-law (3). Suppose the following assumption is verified:

(a) any admissible field of stimulus $\mathbf{s}'$ is such that for any effective response field **r**, eqn (1) holds.

Then any effective stimulus field **s** minimizes the average potential among the admissible fields $\mathbf{s}'$ with $\bar{\mathbf{s}}' = \bar{\mathbf{s}}$:

$$\overline{u(\mathbf{s})} \leqslant \overline{u(\mathbf{s}')} \quad \text{if } \bar{\mathbf{s}}' = \bar{\mathbf{s}} \quad \text{and } \mathbf{s}' \text{ is admissible}, \tag{4}$$

(the proof is left to Appendix 1). It was proved by Arminjon (1991a), pp. 42–43, that

assumption (a) together with the existence of the convex potential u (3) also imply eqn (2) defining a convex macro-potential U (provided that the macro-response $\mathbf{R}$ depends continuously on the macro-stimulus $\mathbf{S}$; in this previous work, the average operator is that defined below, eqn (12), instead of the volume average, but this is immaterial).

Equations (4) and (2) represent a general variational principle for homogenization problems and a general form of the transmission of a convex potential from the micro- to the macro-scale, respectively. If the set S of admissible fields $\mathbf{s}'$ of the micro-stimulus is known, they provide a practical way to determine the effective field $\mathbf{s}$ as a function of the macro-stimulus $\mathbf{S}$ (by solving the minimum problem (4) which has a unique solution if u is *strictly* convex), as well as to compute the macro-law (2). As observed by Suquet (1982, 1987), the definition of the set S contains the boundary conditions at the surface of the RVE Ω, and these are in general unknown since "equivalent macro-elements in a test specimen are constrained by one another, not by the apparatus" (Hill, 1984). In order to obtain actual calculations, the simplifying assumption of uniform surface data $\mathbf{s}_{/\partial\Omega} = \mathbf{S}$ is often adopted (or, to be precise, the weaker assumption $\mathbf{t}\cdot\mathbf{n} = \mathbf{T}\cdot\mathbf{n}$ with $\mathbf{T} = \bar{\mathbf{t}}$ and $\mathbf{n}$ the exterior normal to the boundary $\partial\Omega$, when $\mathbf{s}$ is the stress $\mathbf{t}$; or $\mathbf{v}_{/\partial\Omega} = \mathbf{D}\cdot\mathbf{x}$ with $\mathbf{D} = \bar{\mathbf{d}}$, when $\mathbf{s} = \mathbf{d} = \mathrm{sym}\,(\mathbf{grad}(\mathbf{v}))$ is the strain-rate). It is difficult to estimate the discrepancy caused by this assumption, although it is certainly more justified if very large samples (ideally, infinite ones!) are considered (Suquet, 1982). Thus, when applied to an inhomogeneous material with random structure, the method leads rapidly to huge calculations and is not tractable. In the HTPM on the other hand, periodic boundary conditions are prescribed on the *elementary cell* ω of the periodic medium; this leads to a rigorous and efficient formulation of the homogenization problem (see Section 3.2 for an application).

The statistical theory proposed by Arminjon (1991a) and summarized hereafter accounts for the fact that for statistically homogeneous materials, the relevant information as well as the operative procedures are expressed in terms of the local "state" of the material rather than the local position, leading to a considerable gain in the involved computations. Despite the promising results of Hill and Mandel, only the crude approximations of uniform strain or uniform stress have been able to capitalize on the power of variational principles in a homogenization procedure for general random media with non-linear behavior; moreover, the fact that these approximations give an upper and a lower bound of the macro-potential, respectively, had to be checked for each specific behavior. With the new statistical theory, the classical Voigt–Reuss bounds of linear elasticity are extended to any statistically homogeneous medium with convex local potential. The proposed variational model, which is formulated within this new statistical framework, establishes a continuous transition between the upper and lower bound models.

2.2. *An operational definition of statistical homogeneity*

In the classical theory of *random* media, the local properties of the material and all the relevant fields depend not only on the micro-position $\mathbf{x}$ but also on the "configuration". Each configuration would physically correspond to a given specimen, and the aim is to allow statistics on different specimens. As explained by Kröner (1986), the aim of a truly statistical theory is to calculate an "expectation value" $\langle X\rangle$ of the outcome X of an experiment. The value $\langle X\rangle$ should be the average value over a large number of experiments performed with different specimens which are prepared alike but differ in microscopic details. Formally, the expectation $\langle X\rangle$ is defined as the "ensemble average", the very existence of which is the expression of statistical homogeneity in the usual theory. Then the "ergodic hypothesis" allows to consider $\langle X\rangle$ as equal to a volume average (performed in a given specimen), provided the considered volume is large enough (Beran, 1968; Kröner, 1986). These concepts are not easy to handle and in some cases lead to mathematical difficulties, as observed by Mazilu (1981).

In the proposed theory, the local properties of the material depend only on the local state $\mathbf{X}$, which in turn is a deterministic function of the local position: $\mathbf{X} = \mathbf{F}(\mathbf{x})$. What we call "state" is simply a set of descriptive parameters containing that information on the local behavior and geometry which is considered relevant. Thus, the definition of the state variable $\mathbf{X}$ depends on the material, the studied behavior and the sophistication of the

model in view; e.g. in the case of a mortar reinforced by steel fibres, two successive definitions will be adopted hereafter: (i) $\mathbf{X}$ determines which phase is present at point $\mathbf{x}$, it is thus essentially an integer, say 1 for mortar and 2 for steel; (ii) $\mathbf{X}$ determines the orientation of a cell of a given geometry, containing one steel fibre and the half of two others, embedded in the mortar matrix, in the same volume ratio as the overall material: hence $\mathbf{X}$ is a rotation $\mathbf{g}$ in this refined description (Fig. 1). Note that the ensemble average, as an additive notion, cannot be defined for a rotation variable (unless only rotations all having the same axis are considered, as is the case in the considered example). For convenience, the material is assumed to fill the space. In any finite sample Ω, the *volume distribution of the states* is a probability law P_Ω on the space of states E. It gives the volume fraction of the points $\mathbf{x}$ in Ω for which the state $\mathbf{X}$ belongs to an arbitrary subdomain A of the space of states E:

$$P_\Omega(A) = P_\Omega\{\mathbf{X} \in A\} = V(\{\mathbf{x} \in \Omega; \mathbf{F}(\mathbf{x}) = \mathbf{X} \in A\})/V(\Omega). \tag{5}$$

Here, it may be useful to think of concrete examples, such as a single-phased polycrystal for which, in a first approximation, the local behavior depends only on the local orientation and E is the rotation group. In fact, P_Ω is precisely what can be "measured" (with some data treatment) by the techniques of X-ray or neutron diffraction, quantitative metallography, etc. There are two different cases: P_Ω may either have a density f_Ω, if the state depends continuously on the position $\mathbf{x}$ (except perhaps at the boundaries of constituents); the material is then said to be a *continuum*. Or it may be a discrete law, i.e. an average of Dirac measures, in the case of an *aggregate* where the state is a piecewise constant function of $\mathbf{x}$ (this latter case corresponds to the usual picture of a polycrystal with well-oriented crystallites, but a heavily deformed polycrystal is more likely to be a continuum). The probabilities of higher order that measure the spatial correlation between the states at different points are also easy to define. However, it was shown by Arminjon (1991a) that the consideration of the probabilities of order $n > 1$ of a given state variable $\mathbf{X}$ is in practice equivalent to considering the volume fractions for a "complex" state variable $\mathbf{X}^{(n)}$, the definition of which also contains morphological and topological parameters. Thus, in the new statistical theory, a "volume-fraction model" such as the upper and lower bound models, the self-consistent models or the proposed variational model, may be used so that it actually accounts for spatial information: this will be the case here for the reinforced mortar, when the definition (ii) above is adopted for the state (Section 3.3).

A volume element or "sample" Ω is called *representative* if the law P_Ω is "close" to the law $P_{\Omega'}$ corresponding to any sufficiently large sample Ω'; e.g. in a continuum the average difference between the densities f_Ω and $f_{\Omega'}$ has to be small (lower than some tolerance ε). The volume distribution of the states is said to be *statistically homogeneous* (SH) if arbitrarily representative samples can be found in the material (in practice, one has to be satisfied with a "good" representativity $\varepsilon \ll 1$). Then the laws $P_{\Omega'}$ tend towards a limit distribution of the states, also a probability law P, as the size of the considered sample Ω' infinitely increases; this means in practice that the difference between the laws $P_{\Omega'}$ and $P_{\Omega''}$ should be negligible if Ω' and Ω'' are large enough. We thus get a rigorous, but at the same time an operative definition of a RVE: a given volume element Ω can only be said to be representative *to a certain accuracy* ε. In general, the limit law P has a density, even if the material is a perfect

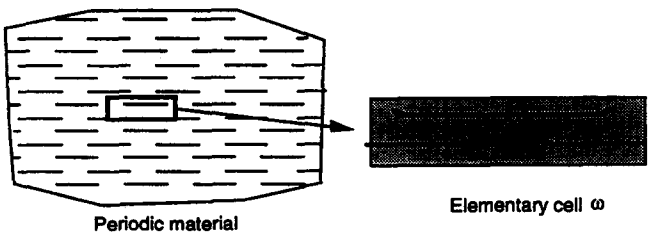

Fig. 1. The elementary cell of the model periodic material with three steel fibres embedded in the cement.

aggregate. This is the formal expression of the well-known fact that a dense histogram is in practice equivalent to a continuous density. Thus there is no essential difference between aggregates and continua, as regards the homogenization problems. An important result is that any *periodic* medium is SH in the proposed sense.

Since the aim of the theory is to transport the homogenization problem from the ("large") physical space to the ("small") space of states, one must also define the notion of a *SH field*, indicating that "on an average", the local value of the field depends only on the local state. To this end, a "state-averaged" function σ^{Ω}, depending on the state variable $\mathbf{X}$, is associated with the considered field $\mathbf{s}$, in any sample Ω. It is obtained by taking the volume average of the field $\mathbf{s}$ in that part of Ω where the state has a given value $\mathbf{X}$:

$$\boldsymbol{\sigma}^{\Omega}(\mathbf{X}) = \left(\int_{\Omega_{\mathbf{X}}} \mathbf{s}\, dV\right) \Big/ V(\Omega_{\mathbf{X}}), \quad \Omega_{\mathbf{X}} = \{\mathbf{x} \in \Omega; \mathbf{F}(\mathbf{x}) = \mathbf{X}\}. \tag{6}$$

(This is the correct definition for an aggregate; for a continuum one takes the average over the points where the state is not farther than a small number d from $\mathbf{X}$, and ideally one makes d tend toward zero.) Thus hereafter, with the definition (i) of the state (the identification number of the considered phase), one simply defines the average values of the strain-rate and stress tensors either in the mortar or in the steel, in the considered part Ω of the material; with the definition (ii) (the state as a rotation), the average is taken over those points of the part Ω, whether the point is in one phase or in the other one, for which the orientation of the nearest steel fiber has a prescribed value $\mathbf{g}$. The field $\mathbf{s}$ is said to be SH if the average difference between the state-averaged functions σ^{Ω} and $\sigma^{\Omega'}$ becomes negligible when Ω and Ω' are sufficiently large. The average difference is calculated with the "limit" volume distribution of the states, P, in other words the difference $(\sigma^{\Omega}(\mathbf{X}) - \sigma^{\Omega'}(\mathbf{X}))$ is weighted by the volume fraction of the state $\mathbf{X}$. If the field $\mathbf{s}$ is SH, one can define *one* state-averaged function σ, independently of a particular sample Ω, as the limit of the function σ^{Ω} as the size of the sample increases. In brief, a field $\mathbf{s}$ is SH when its mean value $\sigma(\mathbf{X})$ in the constituents having a given state $\mathbf{X}$ depends negligibly on the considered RVE. Thus in the reinforced mortar, one can define the average stress $\mathbf{T}^1$ in mortar and $\mathbf{T}^2$ in the steel hooks (with the state as the phase identification number); and one can define the average stress $\tau(\mathbf{g})$ in the composite cells (Fig. 1) having the orientation $\mathbf{g}$ (with the state as the cell orientation $\mathbf{g}$).

2.3. *The no-correlation condition for state-averaged fields*

Whereas the theoretical analysis of micro–macro transition mostly refers to the relevant fields (stress, etc.) as depending on the microscopic position $\mathbf{x}$, the operative procedures have to treat these fields as functions of the local state $\mathbf{X} = \mathbf{F}(\mathbf{x})$, for two main reasons: (1) the existing information on the inhomogeneous microstructure is in the form of distribution statistics of the descriptive parameters $\mathbf{X}$, e.g. volume fractions of the crystals, fiber orientations, etc.; (2) the set of possible states is generally "small" since the same values repeat randomly but endlessly in a SH medium; hence the calculations are shorter in the space of states than in the physical space. Thus one has to assume that "on an average" the local behavior depends only on the local state; hence in the case where the behavior derives from a potential the state-averaged stimulus and response fields σ and ρ must satisfy:

$$\rho(\mathbf{X}) = \frac{\partial u}{\partial \mathbf{s}}(\sigma(\mathbf{X}), \mathbf{X}) = \mathbf{h}(\sigma(\mathbf{X}), \mathbf{X}), \tag{7}$$

which is the statistical counterpart of eqn (3). Note that eqn (7) does not imply that the true local values of the fields, $\mathbf{s}(\mathbf{x})$ and $\mathbf{r}(\mathbf{x})$, depend only on the local state $\mathbf{X} = \mathbf{F}(\mathbf{x})$ (which would mean an unphysical independence on the neighbouring states). We emphasize also that *all* operative procedures for randomly inhomogeneous media implicitly use this assumption.

Just as with the existence of a macroscopic relation between the global averages $\mathbf{S} = \bar{\mathbf{s}}$ and $\mathbf{R} = \bar{\mathbf{r}}$, the relation (7) between the state-averages (6) implies a special kind of macro-homogeneity. To see this, consider the simplest but also the most severe case of an aggregate in which the constituents are geometrically defined, in the sense that the domain $\Omega_{\mathbf{X}}$ where the state is $\mathbf{X}$ is a simply connected one, like a grain in a polycrystal. Then eqn (7) means that the volume averages of $\mathbf{s}$ and $\mathbf{r}$ in $\Omega_{\mathbf{X}}$ are bound by a constitutive relation in the ordinary sense, hence the analysis of macro-homogeneity by Hill (1967, 1984) should apply as it stands—only that here the considered material is the "grain" $\Omega_{\mathbf{X}}$. Thus we know that the assumption of an equation like (7) makes sense only if for any pair of effective stimulus and response fields $\mathbf{s}$ and $\mathbf{r}'$ ($\mathbf{r}'$ being associated with a stimulus field $\mathbf{s}'$, not necessarily the same $\mathbf{s}$) we have the (approximate) no-correlation condition:

$$\int_{\Omega_{\mathbf{X}}} \mathbf{s}:\mathbf{r}' \frac{\mathrm{d}V}{V(\Omega_{\mathbf{X}})} - \left(\int_{\Omega_{\mathbf{X}}} \mathbf{s} \frac{\mathrm{d}V}{V(\Omega_{\mathbf{X}})}\right):\left(\int_{\Omega_{\mathbf{X}}} \mathbf{r}' \frac{\mathrm{d}V}{V(\Omega_{\mathbf{X}})}\right) = \delta_{\mathbf{s},\mathbf{r}'}(\mathbf{X}) \ll \int_{\Omega_{\mathbf{X}}} \mathbf{s}:\mathbf{r}' \frac{\mathrm{d}V}{V(\Omega_{\mathbf{X}})}. \tag{8}$$

This condition is rather restrictive in the present case, since the fields $\mathbf{s}$ and $\mathbf{r}'$ must be defined in the whole aggregate and fulfil eqn (8) simultaneously in every constituent $\Omega_{\mathbf{X}}$ [see Arminjon (1991b) for a detailed discussion]. It means that the fields $\mathbf{s}$ and $\mathbf{r}'$ are "macro-homogeneous" simultaneously *within* every constituent, while the mean value of each field, $\sigma(\mathbf{X})$ and $\rho(\mathbf{X})$, varies from one constituent to its neighbor (this may be called the meso-homogeneity condition). However, one may introduce the weaker condition that the truly macroscopic average of the deviation $\delta_{\mathbf{s},\mathbf{r}'}(\mathbf{X})$ cancels in a RVE Ω:

$$\sum_{\mathbf{X}} \frac{V(\Omega_{\mathbf{X}})}{V(\Omega)} \delta_{\mathbf{s},\mathbf{r}'}(\mathbf{X}) \ll \int_{\Omega} \mathbf{s}:\mathbf{r}' \frac{\mathrm{d}V}{V(\Omega)} = \overline{\mathbf{s}:\mathbf{r}'}. \tag{9}$$

This condition, together with (i) the true macro-homogeneity of $\mathbf{s}$ and $\mathbf{r}'$ expressed by eqn (1) and (ii) the statistical homogeneity of the fields $\mathbf{s}$ and $\mathbf{r}'$ (Section 2.2) implies that state-averaged fields of stimulus and response $\mathbf{s}$ and $\mathbf{r}'$ verify the transported no-correlation condition:

$$\langle \sigma:\rho' \rangle = \langle \sigma \rangle : \langle \rho' \rangle. \tag{10}$$

Here the average operator $\langle\ \rangle$ is defined with the (limit) volume distribution of the states $P(\approx P_{\Omega}$ for a RVE $\Omega)$:

$$\langle \sigma \rangle \equiv \int_E \sigma(\mathbf{X})\, \mathrm{d}P(\mathbf{X}) = \int_E \sigma(\mathbf{X})\, f(\mathbf{X})\, \mathrm{d}\mathbf{X}, \tag{11}$$

with f the density of P and E the space of states: in the case of orientation-dependent behavior, E is the rotation group and f is the orientation distribution function (Bunge, 1982). The definitions imply that:

$$\langle \sigma \rangle = \bar{\mathbf{s}} = \mathbf{S}, \quad \langle \rho' \rangle = \overline{\mathbf{r}'} = \mathbf{R}', \tag{12}$$

but the fundamental eqn (11) still does not follow from the usual no-correlation (1), even for SH fields, for it is not true in general that $\langle \sigma:\rho' \rangle = \overline{\mathbf{s}:\mathbf{r}'}$ unless eqn (9) is satisfied. These results are rigorously stated and proved by Arminjon (1991a).

2.4. *The inhomogeneous variational model for SH materials*

2.4.1. *Statement and justification of the variational principle.* The general conditions of statistical homogeneity discussed in Sections 2.2 and 2.3 justify to "transport the micro–macro problems on the space of states", i.e. to calculate as if the local behavior would

depend only on the local state, as is actually done in operative models. When the state-averaged constitutive equation derives from a potential [eqn (7)], these statistical conditions allow formulation of a variational model; the model consists in determining the actual distribution of the local stimulus, σ (a function of the local state $\mathbf{X}$) as that one which minimizes the average potential:

$$\bar{u}(\sigma) = \langle u(\sigma) \rangle \equiv \int_E u(\sigma(\mathbf{X}), \mathbf{X})\, \mathrm{d}P(\mathbf{X}) \equiv \int_E u(\sigma(\mathbf{X}), \mathbf{X})\, f(\mathbf{X})\, \mathrm{d}\mathbf{X}, \tag{13}$$

among the distributions σ^* satisfying the following two constraints: of "consistency"

$$\langle \sigma^* \rangle \equiv \int_E \sigma^*(\mathbf{X})\, \mathrm{d}P(\mathbf{X}) = \mathbf{S}, \tag{14}$$

i.e. the average stimulus is the macroscopic one $\mathbf{S}$, and of "proximity",

$$h(\sigma^*) \equiv \left[\int_E |\sigma^*(\mathbf{X}) - \mathbf{S}|^p\, \mathrm{d}P(\mathbf{X}) \right]^{1/p} \leqslant r_0. \tag{15}$$

Here p is a fixed real exponent with $p > 1$, giving bounds to the growth of the local potential $u(\mathbf{s}, \mathbf{X})$ on straight lines $\mathbf{s} = \lambda \mathbf{s}_0$ (Arminjon, 1991a); in practice one may actually take $p = 2$, so that the identity in (15) defines the quadratic average of the inhomogeneity of the local stimulus. Thus the actual distribution σ is stated to be a solution of the minimum problem with double constraint:

$$\bar{u}(\sigma^*) = \text{Min}, \quad \langle \sigma^* \rangle = \mathbf{S} \quad \text{and } h(\sigma^*) \leqslant r_0. \tag{16}$$

As discussed below, $r_0 = r_0(\mathbf{S})$ is in fact the inhomogeneity $h(\sigma)$ of σ itself; in other words the minimum (16) is reached at the boundary of the minimization set (except in one very particular case). The minimum (16) is reached by only one distribution $\sigma = \sigma_s$, when the local potential $u(\mathbf{s}, \mathbf{X})$ [with $\mathbf{s} = \sigma(\mathbf{X})$] happens to be a strictly convex function of $\mathbf{s}$ (for almost every $\mathbf{X} \in E$).

The justification of the model relies basically (i) on a straddle of the actual average potential $U(\mathbf{S}) = \bar{u}(\sigma)$ and (ii) on a principle of minimal inhomogeneity. First, if we define a function of the macro-stimulus $\mathbf{S}$ and the inhomogeneity parameter r by the following minimum:

$$U_r(\mathbf{S}) = \text{Min}\, [\bar{u}(\sigma^*)\,;\, \langle \sigma^* \rangle = \mathbf{S} \quad \text{and } h(\sigma^*) \leqslant r], \tag{17}$$

then the classical Reuss–Voigt–Hill lower and upper bounds of the average potential extend to this very general situation and can be set into the form:

$$U_\infty(\mathbf{S}) \leqslant U(\mathbf{S}) \leqslant U_0(\mathbf{S}). \tag{18}$$

The value U_∞ is obtained by dropping the inequality constraint in the minimum (17), thus $U_\infty \leqslant \bar{u}(\sigma)$ since the actual distribution verifies the remaining constraint: $\langle \sigma \rangle = \mathbf{S}$; but, by definition, the macro-potential $U(\mathbf{S})$ is the average $\bar{u}(\sigma)$ of the local potential for the actual distribution of the local stimulus (Arminjon, 1991a): this is the equivalent of eqn (2) in terms of the state-averaged field $\sigma(\mathbf{X})$ instead of that depending on the micro-position, $\mathbf{s}(\mathbf{x})$. Hence the first inequality (18) is clear; it does correspond to the Reuss lower bound, because U_∞ is reached by at least one distribution σ_∞ and this must be associated with a uniform response ρ_∞ by eqn (7), as proved by Arminjon (1991a). Thus the inequality $U_\infty \leqslant U$ may be written $U_R \leqslant U$, with $R = h(\sigma_\infty) = R(\mathbf{S})$, a finite number. By the definition (17), the value U_0 equals the macro-average $\bar{u}(\mathbf{S})$ corresponding to the uniform stimulus $\sigma_0(\mathbf{X}) \equiv \mathbf{S}$.

The upper bound $U \leqslant U_0$ follows then from the convexity of u and from the no-correlation condition (11) [see Arminjon (1991a)]. Since $U_r(\mathbf{S})$ depends continuously on the number r and since $U_R \leqslant U \leqslant U_0$, there is a value $r_0 = r_0(\mathbf{S})$ such that:

$$U_{r_0}(\mathbf{S}) = \bar{u}(\sigma) = U(\mathbf{S}). \tag{19}$$

Moreover, at fixed $\mathbf{S}$, $U_r(\mathbf{S})$ is a strictly decreasing function of r for $0 \leqslant r \leqslant R$ (this is due to the convexity of this function and to the fact that it is constant for $r \geqslant R$; if there are several distributions σ_∞ giving the minimum $U_\infty(\mathbf{S})$, one selects R as the *smallest* value of $h(\sigma_\infty)$ among them). Since the definition (17) implies that $\bar{u}(\sigma) \geqslant U_{h(\sigma)}(\mathbf{S})$ and since $\bar{u}(\sigma) = U_{r_0}(\mathbf{S})$, we have in particular:

$$h(\sigma) \geqslant r_0(\mathbf{S}). \tag{20}$$

Now it is assumed that in fact $h(\sigma) = r_0(\mathbf{S})$: this implies that σ satisfies both constraints in eqn (16) and is thus, by (19), a solution of the minimum problem (16). Let us show that assuming $h(\sigma) = r_0(\mathbf{S})$ is equivalent to the following *principle of minimal inhomogeneity* (PMI): the actual distribution σ of the stimulus is a solution of the minimum problem:

$$h(\sigma^*) = \text{Min}, \quad \langle \sigma^* \rangle = \mathbf{S} \quad \text{and } \bar{u}(\sigma^*) = U(\mathbf{S}). \tag{21}$$

First, the above way of reasoning shows that $h(\sigma^*) \geqslant r_0$ if $\bar{u}(\sigma^*) = U_{r_0}(= U)$ and $\langle \sigma^* \rangle = \mathbf{S}$; hence, in view of (19), the PMI indeed holds if $h(\sigma) = r_0$. Conversely, the PMI implies that $h(\sigma) = r_0$: in fact, let σ_1 be a solution of the minimum problem (17) with $r = r_0$, thus $h(\sigma_1) \leqslant r_0$. The PMI means, again with (19), that $h(\sigma_1) \geqslant h(\sigma)$ and hence $r_0 \geqslant h(\sigma)$, whence the equality by (20).

The PMI states that the actual distribution σ of the local stimulus minimizes the inhomogeneity among the distributions giving the actual value of the macro-potential. For elastic or rigid–plastic materials, the work or the rate of work, respectively, is a potential for the constitutive equation [see Hill (1987), Arminjon (1988, p. 43) and Arminjon and Bacroix (1991) for the rigid-plastic case]. Hence, we may traduce the PMI in a physical language: *the inhomogeneity arises only to reduce the energy consumption.* This principle is not a direct consequence of the mechanical equilibrium equations, but the counterpart is that it might be used for micro–macro problems of different kinds. It is finally this principle which is underlying the success of the "relaxed Taylor theory" (see Van Houtte (1984) for a review of the latter) and the "inhomogeneous variational model" for polycrystals (Arminjon, 1987; Arminjon and Donadille, 1990). In the relaxed Taylor theory, the effect of an ideal grain morphology is taken into account in assuming that only some components of the microscopic stimulus tensor (the strain-rate) may differ from the macroscopic one. In the general model presented here, this method can be extended in assuming an anisotropic norm when defining the local inhomogeneity $|\sigma(\mathbf{X}) - \mathbf{S}|$ [cf. Arminjon and Donadille (1990)]. A development of this idea is deferred until later.

2.4.2. *Assessment of the inhomogeneity parameter.* The inhomogeneity parameter r_0 in eqn (16) must be assessed by extraneous means, if one wants to use this variational principle so as to solve the statistical homogenization–localization problem [determine the distribution σ of the local stimulus, and the macro-potential $U(\mathbf{S}) = \bar{u}(\sigma)$]. This is not surprising, since the equilibrium equations are directly involved neither in the equivalent principles (16) and (21) nor in the statistical equations (7) and (10). The most important point is that r_0 defines the inhomogeneity of the actual distribution σ. Note that the uniqueness of σ has not been assumed, but that in any case $\bar{u}(\sigma)$ and $h(\sigma)$ are well-defined by the minimum problem (16); if two different solutions σ_1 and σ_2 exist to this problem, they give necessarily the same average potential and the same inhomogeneity. This result does not depend on the PMI: $\bar{u}(\sigma_1)$ is well-defined for it is the value of the searched minimum, and $h(\sigma_1)$ is unique because the minimum is reached at the boundary: in the

general case where the actual potential is above the lower bound ($U_{r_0} > U_\infty$), U_r is a strictly decreasing function of $r \leqslant r_0$, hence any solution of U_{r_0} has exactly the inhomogeneity r_0 (in the special case where $U_{r_0} = U_\infty$, the minimization problem (16) always gives the same value U_∞ for any $r_0 \geqslant R$, where $R = h(\sigma_\infty)$ and σ_∞ is one solution; thus, using in that case the PMI, one retains the solution that has the least inhomogeneity, i.e. one takes R as the smallest value r_0 such that $U_{r_0} = U_\infty$: then, any solution of U_R has the inhomogeneity R).

The physical inhomogeneity $r_0 = h(\sigma)$ may be assessed in three different ways: (i) by direct experimental means, e.g. by grid measurements at the microscopic scale (Allais, 1991); (ii) by theoretical calculations, e.g. by searching numerical solutions in a model periodic medium, the elementary cell of which is somehow represententative of the considered randomly inhomogeneous material; and (iii) by indirect experimental means, i.e. by fitting procedures: the simplest one is to determine $r_0(\mathbf{S})$ so that the solution of (16) gives the best experimental agreement for one given macro-stimulus $\mathbf{S}$, and then to admit that r_0 is the same for any other value $\mathbf{S}'$ (Arminjon, 1987); a more refined procedure may take $r_0(\mathbf{S})$ in the form of an analytical anisotropic function $r_0 = \Psi(\mathbf{S}, \alpha_k)$, just like a phenomenological yield criterion in plasticity: the coefficients α_k can be obtained by fitting the optimal values $r_i(\mathbf{S}_i)$ (each of which giving the best agreement for a given stimulus $\mathbf{S}_i$) to the analytical expression $\Psi(\mathbf{S}, \alpha_k)$. The predicting capacity of the model in a concrete situation depends on the sensitivity of the inhomogeneity $r = h(\sigma)$ as function of both the macro-stimulus $\mathbf{S}$ and the volume distribution of the states P (the sensitivity of r to P determines the extent to which one may extrapolate the r value from a material to a similar one); it depends also on the sensitivity of the solution to the minimization problem (17) on the inhomogeneity parameter r: the weaker these sensitivities, the more predictive is the model. It is, however, worth noting that the model is in fact always a predictive one since it gives the whole distribution of the local stimulus as function of the local state, $\sigma(\mathbf{X})$ for all $\mathbf{X}$, from the mere knowledge of the single number r_0 (assuming the PMI).

3. APPLICATION: FAILURE CRITERION OF A FIBER-REINFORCED MORTAR (FRM)

The fiber-reinforced concretes and mortars are the subject of intense research in the field of civil engineering. As a rule, the fibers, made of steel or some other material with relatively high strength and ductility, are incorporated in the weaker matrix. The mostly expected improvement concerns the resistance to tension, which is very poor in non-reinforced mortar. In connection with this, a structure made of reinforced material should have a better resistance to bending and buckling. The reinforcing effect may occur specifically at the scale of the *structure* rather than that of the *material*, in the sense that in some cases the reinforcement is observed mainly at the stage of strain-localization—because, for operating reasons, the matrix is in such cases weaker in the fiber-reinforced material, but the fibers have the capacity to transmit the forces through macroscopic clefts (Rossi *et al.*, 1989). Moreover, precise strain measurements such as those of Torrenti (1988) show that for such materials, even the first maximum of the applied load [which in this work has been retained to define the load of "plastic ruin", see Chambard (1993)] involves the initiation of strain-localization. Theoretically, this kind of behavior should be better described by a model taking into account strain-hardening and damage, as developed, e.g. by Lemaitre and Mazars (1982); however, a rigid–perfectly plastic behavior is assumed here. The reasons are: (i) the rigid–plastic scheme (and with a simple form of the "yield" criterion, see below) gives a correct phenomenological description of the observed maximum loads for materials such as mortars; (ii) the calculations would be much more complex if an evolving behavior was assumed. Thus we focus on the modification of the limit strength of a material with a given matrix, which is due to the adjunction of fibers. In a first step, the simplest definition (the phase number, Section 2.2) is adopted for the state $\mathbf{X}$. This means that only the volume fractions and yield criteria of the two components (mortar and steel) are taken into account; thus the morphological and topological effects intervene only by the inhomogeneity parameter r_0 in eqn (17), and the possible sliding at the interface between a fiber and the matrix is neglected. Then the effects of geometry and sliding are taken into account in adopting the definition (ii) for the state $\mathbf{X}$; thus the local state becomes the orientation $\mathbf{g}$ of the

composite cell represented on Fig. 1, relative to the macroscopic test specimen. The plastic behavior of a material consisting of the juxtaposition of identical such cells is obtained by using the HTPM; the behavior of this periodic material depends directly on the effects of geometry and sliding and is anisotropic. The plastic behavior of the isotropic material, involving randomly distributed fibers, is considered to be that of an SH material with local state $\mathbf{X} = \mathbf{g}$, the "local" (state-averaged) behavior of which is the orientation-dependent one of the above-defined periodic material; the orientation distribution is assumed to be isotropic. Only plane stress states are considered.

The experimental procedure, including manufacturing of the FRM and mechanical tests, is described by Chambard (1993).

3.1. *Single-scale homogenization (volume-fraction model)*

The yield criterion of the steel phase is assumed to be the isotropic one of von Mises; for plane stress ($t_{13} = t_{23} = t_{33} = 0$):

$$\tfrac{3}{2}\,tr(\mathbf{t}-\tfrac{1}{3}(tr\,\mathbf{t})\mathbf{I})^2 = t_{11}^2+t_{22}^2+3t_{12}^2-t_{11}t_{22} = a^2, \tag{22}$$

with a the yield (limit) stress in uniaxial tension ($tr\ \mathbf{t}$ is the trace of tensor $\mathbf{t}$ and $\mathbf{I}$ is the identity tensor). Here the stress is the tensile strength.

The "yield" (failure) criterion of mortar phase is taken to be the isotropic one proposed by Stassi d'Alia (1951, 1967), which allows different yield stresses in tension and compression:

$$\tfrac{3}{2}\,tr(\mathbf{t}-\tfrac{1}{3}(tr\,\mathbf{t})\mathbf{I})^2+(\rho-1)b\,tr\,\mathbf{t} = \rho b^2 = \frac{b'^2}{\rho}, \tag{23}$$

with b the "yield" (failure) stress in uniaxial tension and $b' = \rho b$ the "yield" stress in uniaxial compression. Note that von Mises' criterion (22) is recovered as the particular case $\rho = 1$. Here the phase alone determines the local state and the criterion (23) (with the associated normality flow rule) is assumed to hold in the statistical sense defined in Sections 2.2 and 2.3; this means that equations (22) and (23) apply with $\mathbf{t} = \mathbf{T}^2$ and $\mathbf{t} = \mathbf{T}^1$, the average stress in the steel phase and in the mortar phase, respectively. As has been recalled in Section 2.4.1, the rate of work $W = \mathbf{t}:\mathbf{d}$ is a potential for the rigid–plastic constitutive equation:

$$\mathbf{t} = \frac{\partial W}{\partial \mathbf{d}}\ (\text{if } \mathbf{d} = 0). \tag{24}$$

For a yield criterion which is defined by a "shifted" quadratic form: $f(\mathbf{t}) = (\mathbf{t}-\mathbf{t}_0):\mathbf{B}:(\mathbf{t}-\mathbf{t}_0)$ (with $\mathbf{B}$ a fourth-order tensor and $\mathbf{t}_0$ the "shift" stress), the W function has the form (Arminjon and Bacroix, 1991):

$$W(\mathbf{d}) = \mathbf{t}_0:\mathbf{d}+(\mathbf{d}:\mathbf{B}^{-1}:\mathbf{d})^{1/2}.$$

More specifically, for the isotropic criterion (23) in plane stress condition:

$$W(\mathbf{d}) = b\sqrt{2}\left[\sqrt{\rho^2-\rho+1}\,\sqrt{\tfrac{2}{3}(p^2+q)}-\frac{p}{\sqrt{2}}(\rho-1)\right], \tag{25}$$

where $p = d_{11}+d_{22}$, $q = d_{12}-d_{11}d_{22}$. Thus, the minimization problem (17) takes the form:

$$\left\{\begin{array}{c} W(\mathbf{D}) = f_1 W^1(\mathbf{D}^1) + f_2 W^2(\mathbf{D}^2) = \text{Min}, \\ f_1\mathbf{D}^1 + f_2\mathbf{D}^2 = \mathbf{D}\ \text{(consistency)}, \\ h(\mathbf{D}^1, \mathbf{D}^2) \equiv \sqrt{f_1 \|\mathbf{D}^1 - \mathbf{D}\|^2 + f_2 \|\mathbf{D}^n - \mathbf{D}\|^2} \leqslant r\ \text{(proximity)}, \end{array}\right\} \tag{26}$$

with $\mathbf{D}^1$ and $\mathbf{D}^2$ the average strain-rate in mortar and steel, respectively, f_1 and f_2 the relative volume fractions ($f_1 + f_2 = 1$) and W^1 and W^2 the W function for mortar and steel, thus $\rho = 1$ for W^2 in eqn (26); $\|\mathbf{T}\| = (\Sigma_{i,j} T_{i,j}^2)^{1/2}$ is the Euclidean norm of a tensor $\mathbf{T}$.

This problem has been numerically solved (see Appendix 2), for r starting from 0, until the lower bound $W_\infty(\mathbf{D})$ is reached, i.e. until the value R, such that W_r remains constant for $r \geqslant R$, has been found—and this, for a set of macroscopic tensors $\mathbf{D}$ with Euclidean norm $\|\mathbf{D}\| = 1$ (since W is homogeneous with respect to positive multipliers), $D_{13} = D_{23} = 0$ (since $\mathbf{t}$ is a plane stress, $t_{i3} = 0$ for $i = 1$ to 3, and the material is orthotropic; actually it is even isotropic); the component D_{33} does not play a role in W since $t_{33} = 0$. Thus, $\mathbf{D}$ depends on the three parameters D_{11}, D_{22} and D_{12} which are bound by the relation $\|\mathbf{D}\| = 1$, i.e. it depends on two independent parameters. The values of the material parameters are the following: $a = 1200$ MPa, $b = 2$ MPa, $\rho = 10$ as experimentally determined from tension viz., tension and compression tests on the constitutive materials. Note the very strong inhomogeneity: the ratio of the tensile strength of steel to that of mortar is $a/b = 600$. Two volume fractions of steel fibers were experimentally tested for fiber-reinforced mortars, $f_2 = 0.006$ and $f_2 = 0.012$. The highest fibers percentage actually leads to a weaker strength than the lowest; it is known that beyond a certain volume fraction, the strength of fiber-reinforced mortars decreases, and as a consequence the deformation mechanisms ought not be described by a rigid–plastic scheme beyond this limit. For the present material, this critical volume fraction is low, because the fibers are concentrated in regularly spaced parallel planes (Chambard, 1993). Thus, we give the results for $f_2 = 0.006$, but the comparison has been made also for $f_2 = 0.012$ by Chambard (1993) and leads to the same conclusions regarding the predictive capacity of the model.

For any investigated value of r, a yield criterion ϕ_r is hence defined by the constitutive eqn (25) from the knowledge of the work-rate $W_r(\mathbf{D})$ for any $\mathbf{D}$: it is the yield criterion that the composite material would obey if the inhomogeneity parameter $h(\mathbf{D}^1, \mathbf{D}^2)$ [eqn $(26)_3$] would take the same value r for all $\mathbf{D}$. Then a finite-element simulation of the experimentally tested structure (a $30 \times 30 \times 5$ cm^3 plate for the tension and compression test, a $30 \times 5 \times 5$ cm^3 beam for the four-points bending test) is done, assuming that the material obeys the criterion ϕ_r. For the plates, the plane square structure is meshed by $12 \times 12 = 144$ squares, each divided into 4 triangles with linear interpolation inside a triangle. Figure 2 shows the predicted tension load for the plate as function of the unknown

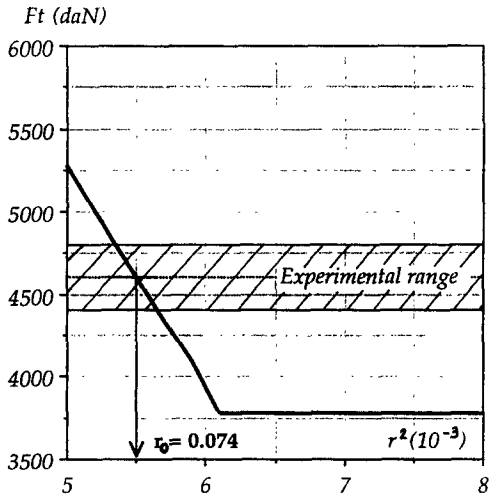

Fig. 2. Predicted tension load vs r^2 [with r the inhomogeneity parameter, eqn (27)] in the case of the volume fraction model, for the random FRM. Comparison with the domain of the obtained experimental values ("experimental range").

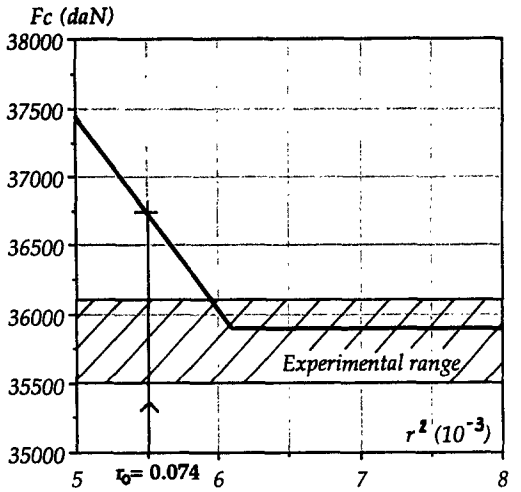

Fig. 3. Predicted tension load vs r^2 [with r the inhomogeneity parameter, eqn (27)] in the case of the volume fraction model, for the compression test. Comparison with the domain of the obtained experimental values ("experimental range").

square r^2, together with the domain of experimental values. A very important difference is found between the lower and upper bounds: $F_{r=R} = 3775$ daN, which corresponds to the strength of the weaker component (the mortar); and $F_{r=0} = 18125$ daN. With the value $r_0 = 0.074$ ($r_0^2 = 5.5 \times 10^{-3}$), the predicted tension load coincides with the center of the experimentally observed limit loads, $F_{exp} = 4600$ daN. The tension test is chosen for the determination of the inhomogeneity parameter r. Thus the value $r_0 = 0.074$ is retained in order to check the predictive capacity of the model when applied with the simplifying assumption that the inhomogeneity parameter does not depend on the macro-stimulus (here **D**).

The predicted loads in compression and bending, as function of r^2, are also compared with the experimental range in Figs 3 and 4, respectively. For these tests also, the predicted loads depend sensitively on the inhomogeneity parameter. With the value $r_0^2 = 5.5 \times 10^{-3}$, adjusted from the tension test, the predicted load falls only slightly over the experimental upper value for bending as well as for compression. It should be noted that each experimental test has been repeated only three–four times, hence the width of the "experimental range" is underestimated. Furthermore, the uncertainty on the input parameters of the model: a, b, ρ in eqns (22) and (23) and f_2 in eqn (26) has not been taken into account. The *experimental agreement* can thus be described as very good, and even as excellent if one

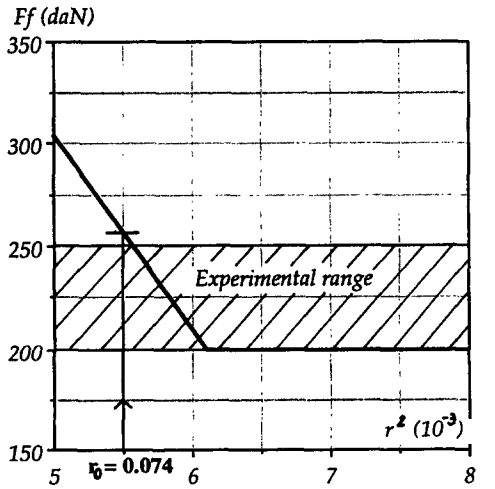

Fig. 4. Predicted tension load vs r^2 [with r the inhomogeneity parameter, eqn (27)] in the case of the volume fraction model, for the 4-points bending test. Comparison with the domain of the obtained experimental values ("experimental range").

compares with the upper and lower bound models: the loads predicted by the upper model ($r = 0$), which do not appear on the figures, are, respectively, $F_0 = 50410$ daN for compression and $F_0 = 1270$ daN for bending. The lower bound is much nearer to experimental values than the upper one, but also distinctly farther from them than the proposed model in the average (with the value r_0 adjusted from tension); for compression, the lower bound model gives a slightly better agreement—but actually, in compression, both predictions are very good. On the other hand, the *predictive* capacity of the model has to be checked in Figs 2 and 3 only, since the tension test has been used to determine the inhomogeneity parameter r_0 ($r_0^2 = 5.5 \times 10^{-3}$). In this regard we see that the reinforcement in bending has been correctly predicted from the reinforcement in tension, and that the reverse prediction (tension from bending) could have been made as well. The observed tiny weakening of the "reinforced" material in compression can obviously not be predicted by any model of the composite involving a rigid–plastic behavior with no sliding. The use of the lower bound model would predict no reinforcement at all and thus must be rejected, although it gives a correct prediction in compression. To make this point clearer, note that the relative reinforcement, defined as $(F_{\text{exp}} - F_{\text{lower bound}})/F_{\text{exp}}$ (with F_{exp} the center of the experimental range), takes the values $+18\%$, $+12.5\%$ and -0.2% for tension, bending and compression, respectively: the weakening in compression is actually negligible. From the 18% reinforcement in tension as an input parameter, the model predicts a 27% reinforcement in bending and a 2.7% reinforcement in compression. It seems doubtful that any other available model of micro–macro transition could do better for this material. The reason why the limit load in compression is (very slightly) lower in the FRM than in the mortar alone may be connected with the experimental observation that clefts propagate tilted through the thickness of the plate in compression, while they propagate in the plane of the plate for tension; hence the steel hooks are likely to play the role of a priming for the clefts in compression.

3.2. *Homogenization in a periodic FRM*

We consider the model periodic material, the elementary cell ω of which is that represented in Fig. 1. The cell ω describes completely the geometrical structure of the periodic material (including the shape of the steel fibers) since the model material is the exact juxtaposition of cells which are identical to ω. The yield criteria (22) and (23) of the two phases are now assumed to hold pointwise inside each phase, but, of course, the stress and strain-rate fields $\mathbf{t}$ and $\mathbf{d}$ are inhomogeneous, also inside one phase. These must be periodic fields. This translates to the following conditions on the boundary $\partial\omega$ of the cell ω with width w and height h, $\omega = [0, w] \times [0, h]$ (Suquet, 1987):

$$\left.\begin{array}{l} \mathbf{t}\cdot\mathbf{n} \text{ opposite on opposite sides of } \partial\omega, \\ \mathbf{v} = \mathbf{D}\cdot\mathbf{x} + \mathbf{v}^*, \mathbf{v} \quad \omega\text{-periodic } [\mathbf{v}^*(x,0) = \mathbf{v}^*(x,h) \quad \text{and} \quad \mathbf{v}^*(0,y) = \mathbf{v}^*(w,y)]. \end{array}\right\} \quad \begin{array}{l} (27) \\ (28) \end{array}$$

Here $\mathbf{n}$ is the outward normal to $\partial\omega$, and $\mathbf{D} = \bar{\mathbf{d}}$ is the volume average of $\mathbf{d}$, in the elementary cell ω and also asymptotically in large samples Ω of the periodic medium. The unknown nature of the contact at the interface between the steel fibers and the mortar has been schematized in three different ways: (i) perfect adhesion (no sliding), (ii) Coulomb friction with $tg\ \phi = 0.5$ or 0.2, (iii) perfect sliding (no friction). Note that the volume-fraction model of Section 3.1 assumes no sliding and is thus not directly comparable with cases (ii) and (iii). As established by Suquet (1982, 1987), the macroscopic yield criterion of the periodic material may be characterized as the boundary of the set of stress tensors $\mathbf{T}$ which are the volume average of an admissible stress field $\mathbf{t}$. The admissibility condition for $\mathbf{t}$ is the conjunction of the boundary condition (27), the equilibrium equation and the plastic admissibility:

$$\phi(\mathbf{t}(\mathbf{x}), \mathbf{x}) \leqslant 1 \quad \text{for all } \mathbf{x} \text{ in } \omega, \tag{29}$$

with ϕ the local yield criterion (here it is defined by eqn (22) or (23), depending on whether $\mathbf{x}$ is in a steel fiber or inside the mortar); for any such field $\mathbf{t}$, the condition $\bar{\mathbf{t}} : \bar{\mathbf{d}}' = \bar{\mathbf{t}} : \bar{\mathbf{d}}'$ is

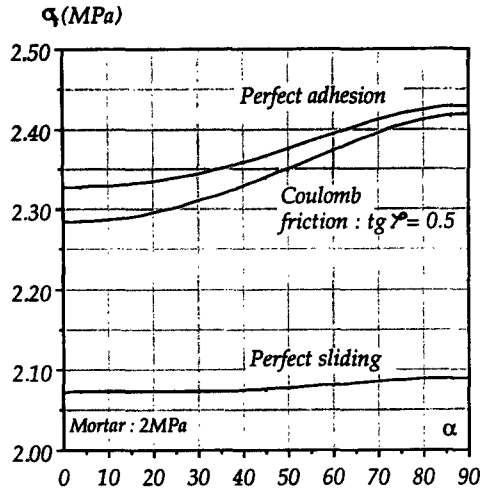

Fig. 5. Predicted tension stress vs fiber orientation α for the model periodic material.

satisfied for all compatible strain-rate field $\mathbf{d}'$ which is deduced from a velocity field $\mathbf{v}'$ satisfying eqn (28). Thus, in the case of standard plasticity for a periodic material, the admissibility condition involves the no-correlation condition (1); hence assumption (a) [see after eqn (3)] is verified, implying the characterization of the macro-potential $W(\mathbf{D}) = \overline{w(\mathbf{d})}$ [eqns (2) and (24)] by the minimum condition (4) which is written here:

$$W_{\text{per}}(\mathbf{D}) = \text{Min}\,[\overline{w(\mathbf{d}')}\,;\mathbf{d}' = e(\mathbf{v}'), \mathbf{v}' = \mathbf{D}\cdot\mathbf{x} + \mathbf{v}^*, \mathbf{v}^*\ \omega\text{-periodic}]. \tag{30}$$

Equation (30) (Suquet, 1982, 1987) is the basis of the numerical calculation of the yield criterion of the periodic material (Turgeman and Pastor, 1987), which uses the FEM with the elementary cell being discretized. In view of the smallness of the fibers, they are schematized as a "generalized 1-D medium" (Salençon, 1983), the strain in fibers is localized at "plastic knee-joints" and another simplified procedure reconciles the 2-D schematization with the small thickness of the fibers; these numerical procedures were proposed and tested by Turgeman (1989).

Once the yield criterion has been obtained, the response of a structure (plate or beam as before) made of the periodic material can be predicted, again by using the FEM in exactly the same way as in Section 3.1. There is no unknown heterogeneity parameter, but now the response of the structure depends on the angle α between the applied load and the fibers. Figures 5 and 6 show the moderate variation of the predicted stress as function of

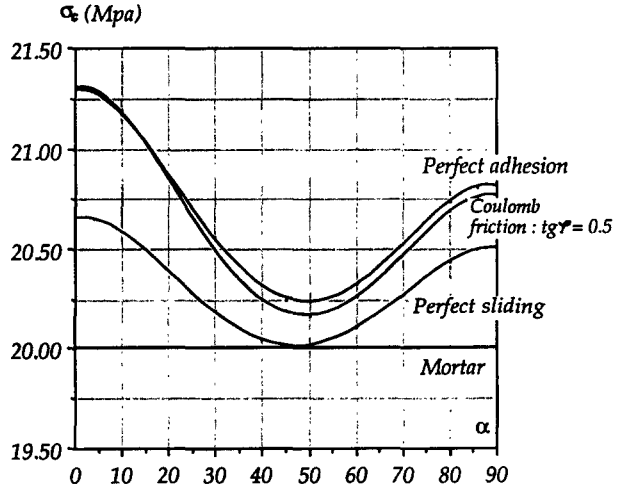

Fig. 6. Predicted tension stress vs fiber orientation α for the compression load.

the fiber orientation α in the periodic FRM, in tension and compression, respectively, and with the different schematizations of the contact at the interface between steel and mortar. This (theoretical) anisotropy is purely an effect of the spatial distribution, since the constituents are isotropic. Depending on the friction coefficient, the predicted limit stress may be lower than that of the matrix for the orientations around $\alpha = 45°$. Experimental tests are currently being performed on the periodic material modelled here.

3.3. *Double-scale homogenization in the random FRM*

We again study the FRM with fibers which are randomly distributed in the Oxy plane (the plane of stress) and we account for geometrical effects and sliding at the interface mortar–fiber, in the following way. The "microscopic behavior" of an inhomogeneous material, i.e. the relationship, depending on the micro-position $\mathbf{x}$, between local stimulus and response $\mathbf{s}(\mathbf{x})$ and $\mathbf{r}(\mathbf{x})$, has to be defined as follows: a microscopic volume element around $\mathbf{x}$ is conceptually taken out of the surrounding material, and considered as a small piece of a *homogeneous* material; this is most rigorously done if the material is an aggregate, having geometrically defined constituents. Now, consider the FRM as an aggregate, its constituents $\omega_1, \omega_2, \ldots$ being arbitrarily defined by a cubic lattice, so that any cubic cell contains in general more than one steel fiber, together with the mortar matrix (Fig. 7). It is clear from the above considerations that the "microscopic behavior" of each cell or constituent ω_i can be rigorously defined and obtained by the HTPM since this latter method describes the behavior of a periodic material consisting of all the same cells ω_i. However, the cells $\omega_1, \omega_2, \omega_3 \ldots$ of the real FRM are not the same, so the behavior obtained by the HTPM will depend on the geometrical and material parameters which characterize the studied cell. In the case of the rigid–plastic FRM, the material parameters [a, b, ρ in (22) and (23), and the friction angle ϕ in Coulomb law at the interface mortar–steel] do not depend on the cell, but the geometrical parameters (the numbers of fibers involved, the position of their centers relative to the cell, and their orientation) do depend on the cell.

Thus, even in much more general situations, one has the theoretical possibility of defining a "complex" state variable $\mathbf{X}$, containing the descriptive (geometrical and material) parameters of the considered cell (Arminjon, 1991a). The HTPM can be used to obtain the local constitutive equation $\mathbf{r} = \mathbf{h}(\mathbf{s}, \mathbf{X})$: in general, one will have to proceed numerically and for each one $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of a set that is representative of the values which the "complex" local state $\mathbf{X}$ can take (i.e. for a representative set of the possible cells). In other words, one

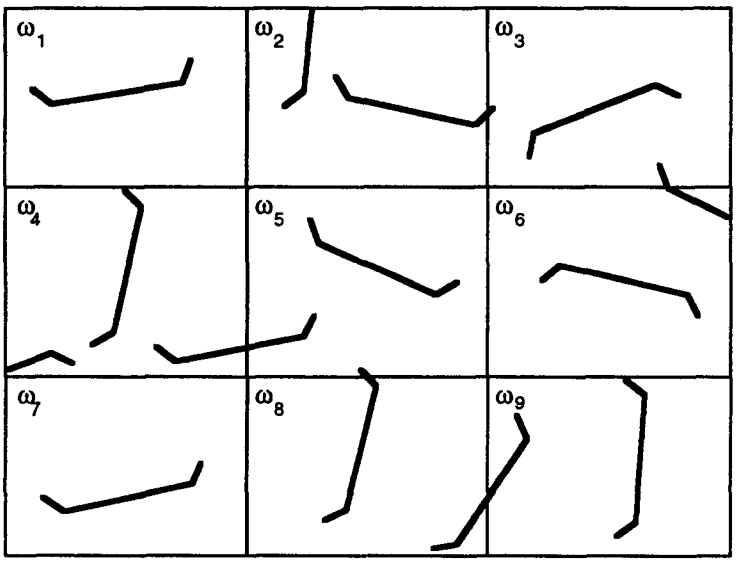

Fig. 7. The random composite, seen as an aggregate.

first solves a series of homogenization problems for periodic media, each one from the microscopic scale to the scale of a given cell. One then has a second homogenization problem, from the level of the different cells to the macroscopic scale; this last problem falls into the application range of the various volume-fraction models for randomly inhomogeneous materials. The proposed variational model is particularly well-suited to treat this if the local behavior derives from a potential [note that if the microscopic (smallest-scale) behavior derives from a potential, that of each cell will also derive from a potential; this has been proved by Suquet (1982, 1987)]; indeed, the minimization problem with double constraint, which is defined for a discrete set $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ of the possible states by an equation like eqn (26) above [though eqn (26) is for $n = 2$], has a general structure which may be applied as it stands, once the local potential is known (App. 2). Of course, the *joined use* of the HTPM (for the first homogenization problem) and our variational volume-fraction model (for the second homogenization problem) is not a volume-fraction model any more, and we call this the "*refined model*".

Here, in order to have a simple illustration, the random variation in the relative position of neighbor fibers is neglected, so that the orientation $\mathbf{g}$ of one fiber with respect to the sample axes defines completely the state $\mathbf{X}$, i.e. the cell. Moreover, the considered cell is that of Fig. 1, in which the three involved fibers have the same orientation. This is not optimal for the random FRM (it would have been more appropriate to take three fibers at, say, 120° from one another), but it has the advantage to describe correctly the experimental periodic FRM (Section 3.2). The variational model described in Section 2.4 is used to predict the rigid-plastic behavior of the random FRM, modelled as an SH aggregate of cells of identical geometry (Fig. 1), but randomly rotated by an angle α around the normal Oz to the plane Oxy. A number $n = 12$ of orientations has been considered to be largely sufficient, in view of the moderate anisotropy predicted for the periodic material (Figs 5 and 6); thus the angles $\alpha_k = (k-1)\Pi/12$ $(1 \leqslant k \leqslant 12)$ have been considered. The minimization problem (17) is written here:

$$W(\mathbf{D}) = \frac{1}{n}\sum_{k=1}^{n} W'_{\text{per}}(\mathbf{D}^k, \alpha_k) = \text{Min},$$

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{D}^k = \mathbf{D}, \quad h((\mathbf{D}^k)) \equiv \sqrt{\frac{1}{n}\sum_{k=1}^{n} \|\mathbf{D}^k - \mathbf{D}\|^2} \leqslant r \tag{31}$$

where $W'_{\text{per}}(\mathbf{d}, \alpha)$ is the rate of work W_{per} in the periodic FRM [eqn (30)], rotated by angle α ($W'_{\text{per}}(\mathbf{d}, \alpha) = W_{\text{per}}(\mathbf{P}(\alpha)\mathbf{d}\mathbf{P}(\alpha)^T)$ with $\mathbf{P}(\alpha)$ the rotation of angle α around Oz). Figures 8,

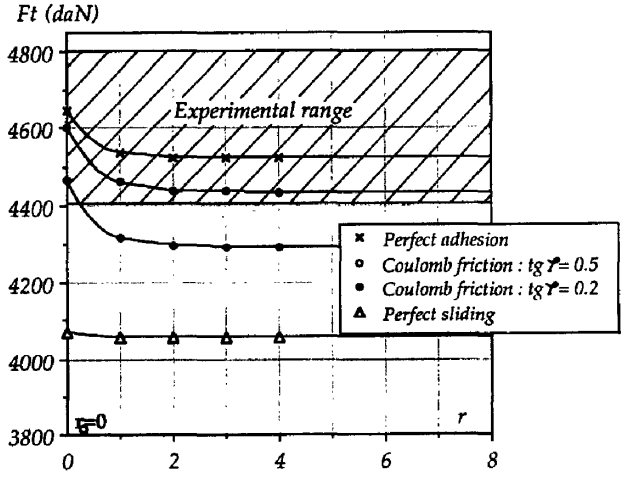

Fig. 8. Predicted tension load vs r [with r the inhomogeneity parameter, eqn (32)], in the case of the "double-scale homogenization" of the random FRC. Comparison with the experimental range.

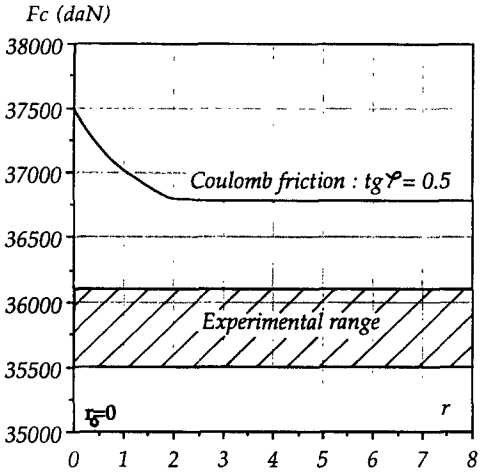

Fig. 9. Predicted tension load vs r [with r the inhomogeneity parameter, eqn (32)], in the case of the "double-scale homogenization" of the compression test. Comparison with the experimental range.

9 and 10 show the predicted loads in tension, compression and bending tests, respectively, as function of the parameter r (unlike Figs 2–4 where the abscissa is r^2) and together with the experimental range. In that case, the Coulomb friction coefficient being assumed to be 0.5, the adjustment of the inhomogeneity parameter from the tension test leads to take $r_0 = 0$, i.e. the upper bound model. Comparison with Figs 2–4 suggests the following comments:

(i) The distance between the upper and lower bounds has been very greatly reduced, e.g. the ratio between the upper and lower bound tension loads passes from 4.58 for the volume-fraction model to 1.04 for the refined model (when the retained value $tg\ \phi = 0.5$ is taken for the friction coefficient at steel–mortar interfaces, which ideally would have to be measured; we note that the French code for reinforced concrete design gives the neighboring value $tg\ \phi = 0.4$). This is natural since a more complete description of the microstructure, including spatial information, has been incorporated. It must be kept in mind, however, that this description of the microstructure applies to an idealized material rather than to the real one: in the real material, the neighboring fibers are randomly oriented relative to each other instead of as shown in the cell of Fig. 1; their mutual distances also are not fixed, neither has the friction coefficient the same value for all fibers (however, it seems reasonable to assume for the discusssion that the value $tg\ \phi = 0.5$ is the actual average of the friction coefficients at all interfaces). The dispersion around the mean values of

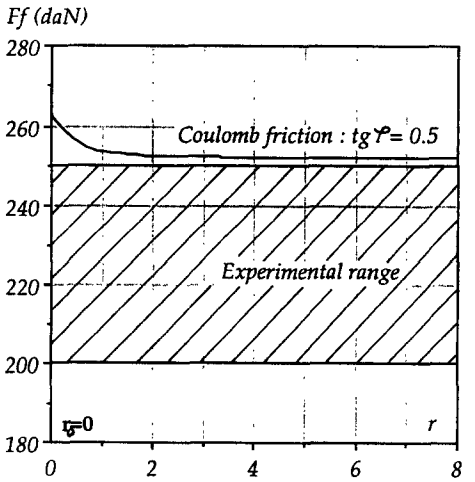

Fig. 10. Predicted tension load vs r [with r the inhomogeneity parameter, eqn (32)], in the case of the "double-scale homogenization" of the 4-points bending test. Comparison with the experimental range.

microstructural parameters could be taken into account, though at an important additional computation cost, by entering more variable parameters into the definition of an ideal cell (i.e. into that of the "complex" state **X**), giving several sets of values to these parameters and running the HTPM for each set. The replacement of the real material by the ideal one, in which only the orientation of the fixed cell of Fig. 1 can vary, implies a too-important narrowing of the upper and lower bounds; in other words, the bounds of the refined model (with given friction coefficient $tg\ \phi = 0.5$) apply to the ideal material but not to the real one, and this explains why.

(ii) The experimental agreement and the predictive capacity of the model are not increased, at least if one of the three mechanical tests is used to make an adjustment of the inhomogeneity parameter, and the measured load in compression is indeed outside the bounds of the refined model. However, if one does not allow to make such an adjustment, and if one assumes the friction coefficient to be $tg\ \phi = 0.5$, then the bounds of the refined model provide a much better assessment of the observed behavior than do the bounds of the volume-fraction model. This even remains true if one does not make any assumption about the nature of the steel–mortar contact, thus taking as "extreme upper bound" of the refined model the upper bound with no-sliding condition and as "extreme lower bound" the lower bound with no-friction condition. The value that could have been taken from the design code for concretes, $tg\ \phi = 0.4$, would probably have given results quite close to the experimental ones.

(iii) The inhomogeneity parameter r takes values which are higher (roughly 20 times) for the refined model than for the volume-fraction model (the comparison can be made, e.g. on the value R that gives the lower bound); this feature also could have been expected, in the sense that the inhomogeneity in the refined model is between the difference cells (Figs 1 and 7) and is thus taken at a smaller scale than in the volume-fraction model, for which r is defined from the difference between the average strain in mortar and in steel. On the other hand, one could also have expected the contrary, for the plastic behaviors of mortar and steel are very different, while the different "cells" all contain the same proportion of both constituents. The numerical experiment contradicts the latter way of reasoning and shows that the scale argument is the right one. This is partly due to the simple fact that, in the volume-fraction model, the steel phase, even though submitted to a strain which is very different from the overall strain, does not play an important role in the inhomogeneity parameter since it is weighted by its small volume fraction $f_2 = 0.006$ [eqn (26)]. In future work, it would be worth investigating the connection between the r parameter as determined here (i.e. so that, for some test, the solution of the variational problem (26) leads to an experimental value: here the load in tension test), and the experimental strain inhomogeneity (which we did not try to measure in this work).

4. CONCLUSION

We started from the recognition of the statistical nature of the no-correlation condition between the fields of stimulus and response (strain and stress, say), which plays the crucial role in micro–macro transition problems occurring in the mechanics of materials; and from the statistical nature of the available information on the microscopic constitution of inhomogeneous materials, too. It seems that the proposed asymptotic definition of statistical homogeneity is more tractable and more operational than the usual one, adapted from theories of statistical physics, and which is based on ergodic theory. It seems important also to emphasize the flexibility of the introduced notion of "state" which is here a set of measurable descriptive parameters (a data-carrier for the local behavior). Different definitions of the state may successively be adopted for the same material: instead of considering probability densities of a higher order to account for spatial information, it is proposed to refine the definition of the state so that it contains morphological and topological parameters describing a "cell", that is, a piece of the material at the microscopic scale. The behavior of a such cell can be obtained by the HTPM; then, the material is considered as an aggregate of different cells and its behavior is modelled from the volume distribution of the "states" characterizing the cells.

To this end, a general variational model may be used, whose input parameters are the volume fractions of the different states and the average inhomogeneity of the local stimulus, i.e. the parameter r. The very nature of this parameter shows that the proposed variational model is ostensibly a "phenomenological micro–macro transition procedure", as opposed to more deterministic approaches starting from a simplified account of the field equations at the microscopic scale. Here the field equations (the mechanical equilibrium equations) intervene only to justify Hill's no-correlation condition. In our opinion, the necessary closure assumptions of models like the self-consistent ones, as well as the extrapolation from an idealized situation (e.g. the elliptic inclusion in the infinite matrix) to a very different real one, make such models more distant from the proper solution of a well-posed mechanical problem than is sometimes believed. The proposed model is based: (i) on a general statistical framework, also necessary to justify other models, and ensuring e.g. that the upper and lower bounds hold; and (ii) on a principle of minimal inhomogeneity, according to which the inhomogeneity in the local stimulus (e.g. strain) occurs only to lower the value of the average potential $\bar{u}$ (consistently with the fact that a homogeneous stimulus gives the upper bound and the lower bound is reached in minimizing $\bar{u}$ without restriction to the homogeneity).

This approach and this model have been applied to predict the failure criterion of a fiber-reinforced mortar from the failure criteria of its components (mortar and steel) and from a simplified description of the geometrical structure of the composite. A rigid–plastic idealization of the mechanical behavior has been adopted. In a first step, only the *volume fractions* are taken into account, which means that the state is defined to be the phase identifier (mortar or steel). With such a schematization, it is not possible to predict the observed reinforcement of the matrix material, because the upper and lower bounds are very distant; however, using the proposed model, the reinforcement in bending can be satisfactorily deduced from that observed in tension, or vice versa. In compression, the composite has a (slightly) lower strength than the matrix; this is out of sight for any rigid–plastic model, but the proposed model indeed predicts a much weaker reinforcement in compression than in tension. In a second step, the *spatial structure* of the composite is assumed to be a random aggregate of cells which differ only by their orientation, each cell containing three steel fibers together with the mortar around them. The behavior of such a cell has been calculated with the HTPM; it is anisotropic and depends on the unknown friction coefficient at the mortar–steel interface. Even with this indeterminacy, the "extreme upper bound" (the upper bound corresponding to perfectly adhesive contact) and the "extreme lower bound" (the lower bound obtained with a nil friction) predicted for the composite, made of randomly oriented cells, are significantly restrained as compared with the volume-fraction model; for a given friction coefficient, the two bounds differ almost negligibly. However, since the refined model assumes an idealized schematization of the real microstructure and the local behavior (e.g. the contact behavior), the experimental agreement of the refined model is not better than that of the volume-fraction model. With regard to the studied composite, it might thus be concluded that the volume-fraction version of the proposed model is sufficient: in order to really improve the micro–macro transition by incorporating spatial elements, one has, of course, to be able to measure the new microstructural parameters that are introduced in doing so. The application of the refined model to this material is worthy as an example of how microgeometry can effectively be taken into account in the proposed approach.

Acknowledgement—We are indebted to Dr Bottero for his help and advice in the experimental part of the work.

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APPENDIX 1: PROOF OF THE MINIMUM (4)

Setting $\mathbf{z} = \mathbf{s}' - \mathbf{s}$, $\phi(\theta) = \bar{u}(\mathbf{s} + \theta\mathbf{z})$ and differentiating the integral defining the average operator [see eqn (2)] with respect to the parameter θ, obtains:

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)_{\theta=0} = \overline{\frac{\partial u}{\partial \mathbf{s}}} : \mathbf{z} = \overline{\mathbf{r} : \mathbf{z}} = \overline{\mathbf{r} : (\mathbf{s}' - \mathbf{s})} = 0, \tag{A1}$$

where the second equality follows from (3) and the last one from (1) and the constraint $\bar{\mathbf{s}}' = \bar{\mathbf{s}}$. The convexity of the local potential u (with respect to $\mathbf{s}$, at given $\mathbf{x}$) is preserved by the average operator, thus $\phi(\theta)$ is a convex function. Equation (A1) implies then that $\phi(0) = \bar{u}(\mathbf{s})$ is the minimum of $\phi(\theta) = \bar{u}[\mathbf{s}+\theta(\mathbf{s}'-\mathbf{s})]$ on the real line; with $\theta = 1$, this proves the minimum condition (4).

APPENDIX 2: NUMERICAL PROCEDURE FOR THE RESEARCH OF THE MINIMA (27) AND (32)

Both problems consist in minimizing a convex functional with following general expression

$$F(\mathbf{Y}) = F(\mathbf{D}^1, \ldots, \mathbf{D}^n) = f_1 W^1(\mathbf{D}^1) + \cdots + f_n W^n(\mathbf{D}^n), \tag{A2}$$

[which is defined on the linear space L of all sequences $\mathbf{Y}$ of n tensors, $\mathbf{Y} = (\mathbf{D}^1, \ldots, \mathbf{D}^n)$], under the following two constraints:

$$A(\mathbf{Y}) = f_1\mathbf{D}^1 + \cdots + f_n\mathbf{D}^n = \mathbf{D}, \tag{A3}$$

and

$$v(\mathbf{Y}) \equiv f_1 \|\mathbf{D}^1 - \mathbf{D}\|^2 + \cdots + f_n \|\mathbf{D}^n - \mathbf{D}\|^2 \leqslant r^2. \tag{A4}$$

Here $\mathbf{D}$ is a given tensor and r, as well as $f_1, \ldots, f_n$ are given positive numbers; except for $W^1, \ldots, W^n$ and $\mathbf{D}^1, \ldots, \mathbf{D}^n$, all superior indices indicate a power. In the example of the paper, the tensors are symmetrical rank-two tensors in the two-dimensional space, thus L is a linear space with dimension $N = 3n$ ($N = 6$ for the volume-fraction model with two phases, $N = 36$ for the refined model with 12 orientations). First, this problem with two convex constraints is replaced by a convex minimum problem without constraint by using the penalty method. Thus we introduce two large numbers p_1 and p_2 and we search for the minimum of the following functional defined on the same space L, without any restriction to the range of $\mathbf{Y} \in L$:

$$G(\mathbf{Y}) = F(\mathbf{Y}) + p_1 \|A(\mathbf{Y}) - \mathbf{D}\|^2 + p_2\{[v(\mathbf{Y}) - r^2]^+\}^2, \tag{A5}$$

where $(f)^+ = f$ if $f \geqslant 0$ and $(f)^+ = 0$ if $f \leqslant 0$. This minimum problem can then be solved by standard optimization techniques which are described e.g. by Minoux (1983). In this work, the well-known "BFGS algorithm" was used. The adjustment of the penalty coefficients p_1 and p_2 has to be done with care; essentially, it must be checked that in some domain on a logarithmic scale for p_1 and p_2, the numerically found minimum of (A5) satisfies both constraints (A3) and (A4) up to a negligible error and remains practically independent of the value of p_1 and p_2. For the studied problems it was found that $p_1 = p_2$ of the order of 10^6 is satisfactory (in general, the values of p_1 and p_2 should be normalized with respect to the order of magnitude of the functional).

Recall that the lower bound should be obtained if one sets r to any value greater than some number R in the constraint (A4), or if one simply does not take this constraint into account; it has been checked that the minimum of the functional (A5) remains indeed unchanged when r is greater than some value. It has also been checked that, unless r is large enough so that the lower bound is reached, the inequality constraint (A4) can be replaced by the corresponding equality constraint (that is, $(f)^+$ replaced by f in (A5)), which confirms that the minimum is reached for a "distribution of the local stimulus", $\mathbf{Y}_0 = (\mathbf{D}_0^1, \ldots, \mathbf{D}_0^n)$ which has exactly the maximum allowed inhomogenity. Finally, it has been checked that for the lower bound solution, the corresponding distribution of the local response (here the set of the associated stress tensors, $\mathbf{Z}_0 = (T_0^1, \ldots, T_0^n)$ with $\mathbf{T}_0^k = (\partial W^k/\partial \mathbf{D}_0^k)$ for $1 \leqslant k \leqslant n$) is uniform, i.e. the tensors $\sigma_0^1, \ldots, \sigma_0^n$ are all the same.